Indian Statistical Institute, Bangalore
M. Math.
First Year, First Semester
Measure Theoretic Probability

Mid-term Examination — Date : September 28, 2011
Maximum marks: 100 — Time: 3 hours

1. Give examples of the following:

   (i) A field which is not a σ-field;

   (ii) A field which is not a montone class;

   (iii) A monotone class which is not a field.

   [10]

2. Show that the smallest monotone class containing a given field is same as the smallest σ-field containing that field. [15]

3. Let $(\Omega, \mathcal{F})$ be a measurable space. Suppose $B_i, 1 \leq i \leq n$ is a collection of mutually disjoint elements in $\mathcal{F}$ such that $\Omega = \bigcup_{i=1}^{n} B_i$. Suppose $f_i, 1 \leq i \leq n$ are real valued measurable functions on Ω (Considering the Borel σ-field on $\mathbb{R}$).Let $f : \Omega \to \mathbb{R}$, be defined by $f(\omega) = f_i(\omega)$, if $\omega \in B_i$, $1 \leq i \leq n$. Show that f is measurable. [15]

4. Let $g_1, g_2, \ldots$ be a sequence of real valued measurable functions on a measurable space $(\Omega, \mathcal{F})$. Suppose $\lim_{n\to\infty} g_n(\omega)$ exists forevery $\omega \in \Omega$. Show that the function g defined on Ω by

$$g(\omega) = \lim_{n\to\infty} g_n(\omega).$$

   is a measurable function. [15]

5. Let $(\Omega_1, \mathcal{F}_1, \mu)$ be a probability space and let $(\Omega_2, \mathcal{F}_2)$ be a measurable space. Suppose $T : \Omega_1 \to \Omega_2$ is a measurable function. Take $\mu_2 = \mu_1 \circ T^{-1}$. Let g be a real valued integrable function on $(\Omega_2, \mathcal{F}_2, \mu_2)$. Now for $A \in \mathcal{F}_2$ show that

$$\int_A g d\mu_2 = \int_{T^{-1}(A)} (g \circ T) d\mu_1.$$

   [20]

[P.T.O.]

6. We have two bags. The first bag has 5 red balls and 3 white balls. The second bag has 4 red balls and 4 white balls. A ball is chosen at random from the first bag and put into the second bag. Now a ball is picked at random from the second bag. (i) What is the probability that the ball picked from the second bag is white? (ii) If the ball picked from the second bag is red what is the conditional probability that the ball chosen from the first bag was also red?

[10]

7. Let X, Y be two independent random variables on a probability space $(\Omega, \mathcal{F}, P)$ with distribution functions F, G. Let Z, W be random variables on this space defined by $Z(\omega) = \text{Max}\ \{X(\omega), Y(\omega)\}$ and $W(\omega) = \text{Min}\ \{X(\omega), Y(\omega)\}$ for ω in Ω. Obtain distribution functions of Z and W. [15].

8. *[Bonus question]* Let $h : [0,1] \to I\!R$ be a bounded Borel measurable function. Define a sequence of functions $\{h_n\}_{n\geq 1}$ by $h_1 = h$ and

$$h_{n+1}(t) = \int_0^t h_n d\mu$$

where μ is the Lebsegue measure. Compute $\lim_{n\to\infty} h_n(t)$ for $t \in [0,1]$. [10]